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Stability of Stochastic Partial Differential Equation

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Stochastic partial differential equations such as occur in vibration problems for mechanical structures subjected to random loading are modelled as infinite dimensional stochastic Itô differential equations using a semigroup approach. Sufficient conditions for exponential stability of the expected energy of the system, as well as for the exponential decay of the sample paths of the displacement and velocity, are given. Under these same conditions it is shown that the zero solution is pathwise asymptotically stable relative to finite dimensional initial conditions. Illustrative examples are included.

1. INTRODUCTION

This paper concerns itself with stochastic stability properties of stochastic second order partial differential equations, such as commonly arise in random vibration models of mechanically flexible systems such as spacecraft, bridges and other mechanical structures. There is already a wide literature on stochastic stability of these systems, usually for a finite dimensional approximation of the system as in [1, 9, 10, 14 and 15], although some analysis of distributed models does exist; see, for example, [14, 17, 21, 22]. The mathematical approach taken here differs from all of these, in that we model the noise using an infinite dimensional Itô integral ([3] or [5]), and so get a "white-noise" description, analogous to that in ordinary stochastic Itô equations reviewed in [14]. This is achieved using a semigroup description of the stochastic partial differential equation as in [2, 5, 6], or [12]. A similar approach was taken by Haussmann [11] to establish stability properties of parabolic-type partial differential equation, although to obtain the sharpest results he used results from [16], where stochastic partial differential equations are studied from a variational approach. Further comparisons between the work of Haussmann [11] and Pardoux [16] and this work are made in the conclusions, Section 6.

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Section 2 contains the mathematical formulation of the model, Section 3 contains a summary of the second order stability properties deducible from known results in [13], Section 4 is devoted to establishing an energy inequality and moment equations and Section 5 establishes the asymptotic stability results.

2. Semigroup Formulation of Second Order Systems

We consider the following second order system with $\alpha > 0$,

$$v_{tt} + \alpha v_t + Av = 0; \qquad v(0) = v_0, \quad v_t(0) = v_1, \tag{2.1}$$

where A is a positive, self adjoint operator on a Hilbert space H and has domain $D(A)$.

Then $\mathscr{H} = D(A^{1/2}) \times H$ is a Hilbert space under the inner product

$$\langle z, \bar{z} \rangle_{\mathscr{H}} = \langle A^{1/2} z_1, A^{1/2} \bar{z}_1 \rangle_H + \langle z_2, \bar{z}_2 \rangle_H, \tag{2.2}$$

where

$$z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \qquad \bar{z} = \begin{pmatrix} \bar{z}_1 \\ \bar{z}_2 \end{pmatrix}.$$

Following [5] or [18], the closed, linear operator on $\mathscr{H}$

$$\mathscr{A}_0 = \begin{pmatrix} 0 & I \\ -A & -\alpha I \end{pmatrix} \qquad \text{with domain } D(A) \times D(A^{1/2}) \tag{2.3}$$

generates a strongly continuous contraction semigroup S_t^0.

Thus (2.1) may be written as a first order differential equation on $\mathscr{H}$,

$$\dot{z} = \mathscr{A}_0 z; \qquad z(0) = z_0, \tag{2.4}$$

where

$$z = \begin{pmatrix} v \\ v_t \end{pmatrix} \qquad \text{and} \qquad z_0 = \begin{pmatrix} v_0 \\ v_1 \end{pmatrix}.$$

We would like to consider the more general class

$$v_{tt} + \alpha v_t + Av + Fv = 0; \qquad v(0) = v_0, \quad v_t(0) = v_1. \tag{2.5}$$

or equivalently on $\mathscr{H}$

$$\dot{z} = \mathscr{A} z; \qquad z(0) = z_0, \tag{2.6}$$

where

$$\mathscr{A} = \mathscr{A}_0 + \begin{pmatrix} 0 & 0 \\ -F & 0 \end{pmatrix}. \tag{2.7}$$

Provided $F \in \mathscr{L}(D(A^{1/2}), H)$, by the perturbation theory for semigroups [9], $\mathscr{A}$ also generates a strongly continuous contraction semigroup.

The following stability result from [18] is used in Section 3.

LEMMA 2.1. *If* $\alpha > 0$ *and* $\boldsymbol{\omega}(-A) = \sup\{\operatorname{Re} \lambda : \lambda \in \sigma(-A)\} < 0$, *then* $\mathscr{A}_0$ *given by* (2.3) *generates an exponentially stable contraction semigroup* S_t^0 *with*

$$\|S_t^0\|_{\mathscr{L}(\mathscr{H})} \leqslant e^{-\omega t},$$

where

$$\omega \geqslant \frac{2\alpha\,|\boldsymbol{\omega}(-A)|}{4\,|\boldsymbol{\omega}(-A)| + \alpha(\alpha + \sqrt{\alpha^2 + 4\,|\boldsymbol{\omega}(-A)|})}.$$

LEMMA 2.2 (from [5]). *If* $\|F\|_{\mathscr{L}(D(A^{1/2}),H)} \leqslant m < \omega$, *then* $\mathscr{A}$ *given by* (2.7) *also generates an exponentially stable contraction semigroup* S_t *with*

$$\|S_t\|_{\mathscr{L}(H)} \leqslant e^{-(\omega - m)t}.$$

For other results on exponential stability of semigroups see [18–20].

Formally, the class of stochastic systems we wish to consider is

$$v_{tt} + \alpha v_t + (A + F)\,v + D_1 \xi_1 + D_2 v \xi_2 + D_3 v_t \xi_3 = 0, \tag{2.8}$$

where ξ_1, ξ_2 and ξ_3 are mutually independent "white-noise" processes, F and $D_2 \in \mathscr{L}(D(A^{1/2}), H)$, $D_3 \in \mathscr{L}(H)$ and $D_2 \in \mathscr{L}(H, K)$, where K is another Hilbert space. This means that ξ_2 and ξ_3 are real valued and spatially independent, but ξ_3 can be spatially dependent.

One usually models "white-noise" disturbances by Itô stochastic integrals with respect to a Wiener process and following the approach in [4] or [5] we are led to consider the formal stochastic evolution equation

$$dz = \mathscr{A} z\, dt + \begin{pmatrix} 0 \\ D_1 \end{pmatrix} dw(t) + \begin{pmatrix} 0 & 0 \\ D_2 & 0 \end{pmatrix} z\, d\beta_1(t)$$

$$+ \begin{pmatrix} 0 & 0 \\ 0 & D_3 \end{pmatrix} z\, d\beta_2(t), \qquad z(0) = z_0, \tag{2.9}$$

where $\mathscr{A}$, D_1, D_2 and D_3 are as before, β_1 and β_2 are mutually independent standard Brownian motions (i.e., with unit variance parameters) and w is a K-valued Wiener process independent of β_1 and β_2 and with covariance operator W. z_0 is a second order random variable on $\mathscr{H}$ and independent of β_1, β_2 and w.

Stochastic integrals with non-random integrals are defined in [3] or [4] and have properties analogous to scalar Itô integrals and so the last three terms in (2.9) are well defined. We could then define a (strong) solution of (2.9) to be a process $z(t)$ which satisfies the integrated version of (2.9) w.p.1. Implicitly, this implies that $z(t) \in D(A)$ w.p.1, but for stochastic evolution equations (even for the simple case with $\beta_1 = \beta_2 = 0$) this does not hold in general (see [5, Examples 5.39, 5.40]). A more useful concept is that of a mild solution, by which we mean a process $z(t) \in C(0, T; L_2(\Omega, \mu; \mathscr{H})$ which satisfies the following integral equation with probability 1:

$$\begin{aligned} z(t) = S_t z_0 &+ \int_0^t S_{t-s} \begin{pmatrix} 0 \\ D_1 \end{pmatrix} dw(s) \\ &+ \int_0^t S_{t-s} \begin{pmatrix} 0 & 0 \\ D_2 & 0 \end{pmatrix} z(s)\, d\beta_1(s) \\ &+ \int_0^t S_{t-s} \begin{pmatrix} 0 & 0 \\ 0 & D_3 \end{pmatrix} z(s)\, d\beta_2(s). \end{aligned} \tag{2.10}$$

We note that (2.9) can be obtained from (2.10) by formal differentiation. Such solutions for $\beta_1 = \beta_2 = 0$ have been examined in [4, 5] and for linear stochastic evolution equations with state dependent noise in [12]. From [12] we see that (2.10) has a unique solution $z(t) \in C(0, T; L_2(\Omega, \mu; \mathscr{H})$ which implies that

$$\sup_{0 \leqslant t \leqslant T} E\{\|z(t)\|^2_{\mathscr{H}}\} < \infty. \tag{2.11}$$

A closer examination of different concepts of solutions for (2.10) is found in [2]. In general mild solutions do not have continuous sample paths, but we can prove that for our class of systems (2.10) this property does hold.

LEMMA 2.3. *$z(t)$ given by (2.10) has continuous sample paths under our assumptions.*

Proof. Consider the special case $\alpha = 0$ and $F = 0$. Then $\bar{\mathscr{A}} = \left(\begin{smallmatrix} 0 & I \\ -A & 0 \end{smallmatrix}\right)$ generates a strongly continuous *group* $\bar{S}_t$. To establish the sample path continuity of (2.10) it suffices to consider a typical term

$$\int_0^t \bar{S}_{t-s} \begin{pmatrix} 0 & 0 \\ D_2 & 0 \end{pmatrix} z(s)\, d\beta_1(s) = \bar{S}_t \int_0^t \bar{S}_{-s} \bar{D}_2 z(s)\, d\beta_1(s), \tag{2.12}$$

with

$$\bar{D}_2 = \begin{pmatrix} 0 & 0 \\ D_2 & 0 \end{pmatrix},$$

which is valid since $\bar{S}_t$ is a group. But from [3], $\int_0^t \Phi(s)\, dw(s)$ has continuous sample paths for a general Wiener process and a stochastic process $\Phi(t)$ adapted to $w(t)$ and with $E \int_0^t \|\Phi(s)\|^2\, ds < \infty$. Equation (2.11) ensures that this result can be applied to $\int_0^t \bar{S}_{-s} \bar{D}_2 z(s)\, d\beta_1(s)$ and so (2.12) has continuous sample paths.

The general case follows by a perturbation argument. Let $B \in \mathscr{L}(\mathscr{H})$; then from [5], $\bar{\mathscr{A}} + B$ generates a strongly continuous semigroup T_t given by

$$T_{t-s}h = \bar{S}_{t-s}h + \int_s^t T_{t-u} B \bar{S}_{u-s} h\, du, \qquad h \in \mathscr{H}. \tag{2.13}$$

We consider a typical term in (2.10),

$$\begin{aligned} &\int_0^t T_{t-s} \bar{D}_2 z(s)\, d\beta_1(s) \\ &\quad = \int_0^t \bar{S}_{t-s} \bar{D}_2 z(s)\, d\beta_1(s) + \int_0^t \int_s^t T_{t-u} B \bar{S}_{u-s} \bar{D}_2 z(s)\, du\, d\beta_1(s) \\ &\quad = \int_0^t \bar{S}_{t-s} \bar{D}_2 z(s)\, d\beta_1(s) + \int_0^t T_{t-u} B \left[\int_0^u \bar{S}_{u-s} \bar{D}_2 z(s)\, d\beta_1(s) \right] du, \end{aligned}$$

by the generalized stochastic Fubini theorem [4]. But from our first part, we know that terms like (2.12) have continuous sample paths and consequently so does $\int_0^t T_{t-s} \bar{D}_2 z(s)\, d\beta_1(s)$, where T_t is any bounded perturbation of $\bar{S}_t$.

3. Second Order Stability Properties

The results in this section are a summary of those established in [13] and are valid for an arbitrary strongly continuous semigroup S_t on a Hilbert space $\mathscr{H}$.

Definition 3.1. *Mean Square Stability.* Consider

$$z(t) = S_t z_0 + \int_0^t S_{t-s} D z(s)\, dw(s), \tag{3.1}$$

where $z_0 \in \mathscr{H}$, $D \in \mathscr{L}(\mathscr{H}, \mathscr{L}(K, \mathscr{H}))$, and $w(t)$ is a Wiener process on a Hilbert space K, with covariance operator W. Equation (3.1) is said to be *mean square stable* if for each $z_0 \in \mathscr{H}$, the mild solution of (3.1) satisfies

$$\int_0^\infty E\{\|z(s)\|_{\mathscr{H}}^2\} d\, ds < \infty. \tag{3.2}$$

Consider

$$z(t) = S_t z_0 + \int_0^t S_{t-s} Dz(s)\, dw(s) + \int_0^t S_{t-s} F\, dw_2(s), \tag{3.3}$$

where S_t, D and w_0 are as above and $F \in \mathscr{L}(K_2, \mathscr{H})$ and w_2 is a Wiener process on a Hilbert space K_2 and is independent of w. Equation (3.3) is said to be *mean square stable* if

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T E\{\|z(s)\|_{\mathscr{H}}^2\}\, ds < \infty. \tag{3.4}$$

THEOREM 3.1. (a) *Equation* (3.1) *is mean square stable if and only if for all* $z_0 \in \mathscr{H}$

$$E\{\|z(t)\|_{\mathscr{H}}^2\} \leqslant Ce^{-\omega t} \|z_0\|_{\mathscr{H}}^2 \tag{3.5}$$

for some C, $\omega > 0$, *or equivalently*:

There exists a self adjoint, nonnegative operator $P \in \mathscr{L}(\mathscr{H})$, *which is a solution of*

$$2\langle \mathscr{A} z, Pz\rangle_{\mathscr{H}} + \langle z, z\rangle_{\mathscr{H}} + \operatorname{trace}[(Dz)^* P\, Dz\, W] = 0 \tag{3.6}$$

for $z \in D(\mathscr{A})$.

(b) *Equation* (3.3) *is mean square stable if and only if* (3.1) *is.*

(c) *A sufficient condition for mean square stability is that* $\mathscr{A}$ *generate an exponentially stable semigroup and*

$$\left\| \int_0^\infty S_t^* \Delta S_t\, dt \right\|_{\mathscr{H}} < 1, \tag{3.7}$$

where $\langle \Delta h, k\rangle_{\mathscr{H}} = \operatorname{trace}[(Dh)^* DkW]$ *for* $h, k \in \mathscr{H}$.

Consequently combining known results on exponential stability for the deterministic equation (2.6) and estimates from (3.7) provides us with a check of mean square stability of the stochastic equation.

EXAMPLE 1. *Lateral Displacement of a Damped Stretched String Subjected to Random Loading* [17].

$$\frac{\partial^2 v}{\partial t^2} + \alpha \frac{\partial v}{\partial t} - \frac{\partial^2 v}{\partial x^2} + (m + c\xi)\frac{\partial v}{\partial x} = 0,$$

$$v(0, t) = 0 = v(1, t),$$

where ξ is a scalar "white noise" process of unit variance and α, m, c are nonrandom constants and $\alpha > 0$. Let

$$H = L_2(0, 1), \qquad A = -\frac{\partial^2}{\partial x^2},$$

$$D(A) = \begin{Bmatrix} u \in H: & u_x, u_{xx} \in H \\ \text{and} & u(0) = 0 = u(1) \end{Bmatrix}.$$

Suppose first that $m = 0$, and take

$$\mathscr{A} = \begin{pmatrix} 0 & I \\ -A & -\alpha I \end{pmatrix}.$$

Then $|\boldsymbol{\omega}(-A)| = \pi^2$ and $\|S_t\| \leqslant e^{-\omega t}$, where

$$\omega \geqslant \frac{2\alpha\pi^2}{4\pi^2 + \alpha(\alpha + \sqrt{\alpha^2 + 4\pi^2})}.$$

Now

$$\left\| \int_0^\infty S_t^* \Delta S_t \, dt \right\|_{\mathscr{H}} \leqslant \int_0^\infty \|S_t\|^2 \|\Delta\| \, dt$$

$$\leqslant \int_0^\infty e^{-2\omega t} c^2 \, dt$$

$$= c^2/2\omega.$$

Thus we have mean square stability if

$$c^2 < \frac{4\alpha\pi^2}{4\pi^2 + \alpha(\alpha + \sqrt{\alpha^2 + 4\pi^2})}.$$

For the case $m \neq 0$, this estimate must be replaced by

$$c^2 < \frac{4\alpha\pi^2}{4\pi^2 + \alpha(\alpha + \sqrt{\alpha^2 + 4\pi^2})} - |m|.$$

EXAMPLE 2. *Vibration of a Panel in Supersonic Flow Subjected to Random End Loads* [14, 17].

$$\frac{\partial^2 v}{\partial t^2} + \alpha \frac{\partial v}{\partial t} + m \frac{\partial v}{\partial x} + (f + c\xi) \frac{\partial^2 v}{\partial x^2} + \frac{\partial^4 v}{\partial x^4} = 0,$$

$$v(0, t) = 0 = v(1, t) = \frac{\partial^2 v}{\partial x^2}(0, t) = \frac{\partial^2 v}{\partial x^2}(1, t),$$

where α, m, f, c are nonrandom constants, $\alpha > 0$ and ξ is a scalar white noise process of unit variance. Let

$$H = L_2(0, 1), \qquad A = \frac{\partial^4}{\partial x^4} + f \frac{\partial^2}{\partial x^2},$$

$$D(A) = \left\{ \begin{matrix} h \in H: & h_x, h_{xx}, h_{xxx}, h_{xxxx} \in H \\ \text{and} & h(0) = 0 = h(1), h_{xx}(0) = 0 = h_{xx}(1) \end{matrix} \right\}.$$

Suppose first that $m = 0$. For A to be self adjoint and positive, we require $f < \pi^2$ and obtain that $|\boldsymbol{\omega}(-A)| = \pi^2(\pi^2 - f^2)$, and $\mathscr{A} = \left(\begin{smallmatrix} 0 & I \\ -A & -\alpha I \end{smallmatrix} \right)$ is stable if $f < \pi^2$ and $\alpha > 0$, with $\|S_t\| \leqslant e^{-\omega t}$ and

$$\omega \geqslant \frac{2\alpha\pi^2(\pi^2 - f^2)}{4\pi^2(\pi^2 - f^2) + \alpha\sqrt{\alpha^2 + 4\pi^2(\pi^2 - f^2)}}.$$

As in the previous example,

$$\left\| \int_0^\infty S_t^* \varDelta S_t \, dt \right\|_{\mathscr{H}} \leqslant c^2/2\omega$$

and sufficient conditions for stability are $c^2 < 2\omega$. For $m \neq 0$ we can obtain sufficient conditions $c^2 < 2\omega - |m|$.

It is clear from these two examples that our stochastic stability estimates rely heavily on good estimates for the exponential stability of S_t and that Lemma 2.2 gives a very crude estimate. There are other approaches, for example, in [18–20], which yield better estimates for the deterministic case.

4. THE ENERGY EQUALITY AND MOMENT EQUATIONS FOR STOCHASTIC SECOND ORDER EQUATIONS

First we need an extension of Itô's lemma from [3] to the case of an unbounded operator, similar to that asserted in [13].

LEMMA 4.1. *Suppose that $z(t)$ is an $\mathscr{H}$-valued stochastic process, which is the strong solution of*

$$dz(t) = \mathscr{A} z(t)\,dt + D(z(t))\,dw(t); \qquad z(0) = z_0 \in D(\mathscr{A}), \tag{4.1}$$

where $\mathscr{A}$ is a strongly continuous semigroup on $\mathscr{H}$, $w(t)$ is a Wiener process on a Hilbert space K with covariance operator W, $D \in \mathscr{L}(\mathscr{H}, \mathscr{L}(K, \mathscr{H}))$ and $E\{\|z_0\|^4\} < \infty$. Then if $\Phi: \mathscr{H} \to R^1$ is twice Fréchet differentiable, the process $y(t) = \Phi(z(t))$ satisfies

$$y(t) = y(0) + \int_0^t (\langle \Phi_z(z(s)), \mathscr{A} z(s)\rangle + \tfrac{1}{2}\,\mathrm{trace}[Dz(s)\,W(Dz(s))^*\,\Phi_{zz}])\,ds$$
$$+ \int_0^t \langle \Phi_z(z(s)), Dz(s)\,dw(s)\rangle. \tag{4.2}$$

Proof. From [10], since $z(t)$ is a strong solution of (4.1), $z(t) \in D(\mathscr{A})$ w.p.1. and $\int_0^t \|\mathscr{A} z(s)\|\,ds < \infty$ w.p.1.

Furthermore, $\sup_{0 \leqslant t \leqslant T} E\{\|z(t)\|^2\} < \infty$. Thus to apply the Itô's lemma of [3], we need only establish $\int_0^t E\{\|z(s)\|^4\}\,ds < \infty$, which we perform in two steps.

(a) Let $\Psi(t)$ be an $\mathscr{L}(K, \mathscr{H})$-valued stochastic process adapted to the sigma fields generated by $w(t)$, such that

$$\int_0^T E\{\|\Psi(t)\|^4\}_{\mathscr{L}(K,\mathscr{H})}\,dt < \infty, \tag{4.3}$$

then

$$E\left\{\left\|\int_0^T \Psi(t)\,dw(t)\right\|^4\right\} \leqslant 3(\mathrm{trace}\,W)^2\,T \int_0^T E\{\|\Psi(t)\|^4\}\,dt. \tag{4.4}$$

Although (to my knowledge) this has not been proved in a published article to date, it is straightforward and can be proved in the style of Proposition 2.9 of [3], that is, by first proving the result for Ψ a step function in t and extending the result to Ψ satisfying (4.3) by approximating Ψ by a sequence of step functions.

(b) The strong solution of (4.1) is also the mild solution of the integral equation [2]

$$z(t) = S_t z_0 + \int_0^t S_{t-s}\,D(z(s))\,dw(s). \tag{4.5}$$

Furthermore, we can prove that $z(t)$ is also the unique solution of (4.5) in Z, the space of $\mathscr{H}$-valued processes, adapted to $w(t)$ under the norm

$$\|z\|_4 = \left(\int_0^T E\{\|z(s)\|_{\mathscr{H}}^4\}\,ds\right)^{1/4}. \tag{4.6}$$

It is sufficient to show that Λ is a contraction on Z, where

$$\Lambda z = \int_0^t S_{t-s}\,D(z(s))\,dw(s). \tag{4.7}$$

If $E\{\|z_0\|^4\} < \infty$, then Λ maps Z into Z by (4.4).

It is now routine to show that Λ is a contraction under the following equivalent norm for sufficiently large $\mu > 0$:

$$\|z\|_\mu = \left(\int_0^T e^{-\mu s} E\{\|z(s)\|_{\mathscr{H}}^4\}\,ds\right)^{1/4}.$$

Thus Λ is also a contraction on Z and $z(t)$ has a unique solution in Z and

$$\int_0^T E\{\|z(s)\|^4\}\,ds < \infty.$$

We wish to obtain an energy equality for the stochastic system (2.10) with $D_1 = 0$, that is, for

$$z(t) = S_t z_0 + \int_0^t S_{t-s}\begin{pmatrix} 0 & 0 \\ D_2 & 0 \end{pmatrix} z(s)\,d\beta_1(s)$$

$$+ \int_0^t S_{t-s}\begin{pmatrix} 0 & 0 \\ 0 & D_3 \end{pmatrix} z(s)\,d\beta_2(s), \tag{4.8}$$

where $\mathscr{A}$ of (2.7) generates the semigroup S_t on $\mathscr{H} = D(A^{1/2}) \times H$. Writing $z(t) = \binom{v(t)}{v_t(t)}$ and using $\Phi(h) = \langle h, h\rangle_{\mathscr{H}}$, we can use (4.2) to obtain a formal expression for the energy of the system, namely,

$$\|z(t)\|_{\mathscr{H}}^2 = \|A^{1/2}v\|_H^2 + \|v_t\|_H^2$$

$$= \|A^{1/2}v_0\|_H^2 + \|v_1\|_H^2 - 2\alpha\int_0^t \|v_s\|^2\,ds$$

$$+ \int_0^t (\|D_2 v\|^2 + \|D_3 v_s\|^2)\,ds - z\int_0^t \langle Fv, v_s\rangle\,ds$$

$$+ 2\int_0^t \langle v_s, D_2 v\rangle\,d\beta_1 + 2\int_0^t \langle v_s, D_3 v_s\rangle\,d\beta_2 \qquad \text{w.p.1.} \tag{4.9}$$

In fact, this is the energy equality we are seeking, but it does not follow from Lemma 4.1 directly, as in this case (4.1) does not have strong solutions. We establish (4.9) for our system by an approximation argument used by Ichikawa in [13].

LEMMA 4.2. *If $\mathscr{A}$ generates a strongly continuous semigroup on $\mathscr{H}$ and $\lambda \in \rho(\mathscr{A})$, $z_0 \in D(\mathscr{A})$* w.p.l., *then the following system has a unique strong solution $z_\lambda(t)$,*

$$dz(t) = \mathscr{A} z(t)\, dt + D_\lambda(z(t))\, dw(t); \qquad z(0) = z_0, \tag{4.10}$$

where $D_\lambda = \lambda R(\lambda, \mathscr{A})\, D$, $R(\lambda, \mathscr{A}) = (\lambda I - \mathscr{A})^{-1}$ and D, w as for (4.1).
Furthermore,

$$\sup_{0 \leqslant t \leqslant T} E\{\|z_\lambda(t) - z(t)\|^2_{\mathscr{H}}\} \to 0 \qquad as \quad \lambda \to \infty, \tag{4.11}$$

where $z(t)$ is the mild solution of (4.1) *with initial condition $z_0 \in D(\mathscr{A})$* w.p.1.

Proof. See [13].

LEMMA 4.3. *Equation* (4.9) *holds* w.p.1. *for $z(t) = \binom{v}{v_t}$, the mild solution of* (2.10) *with $D_1 = 0$.*

Proof. For $z_0 \in D(\mathscr{A})$, (4.11) has the strong solution $z_\lambda(t)$ and so by Lemma 4.1,

$$\begin{aligned} \|z_\lambda(t)\|^2_{\mathscr{H}} = \|A^{1/2} v_0\|^2 + \|v_1\|^2 &+ 2 \int_0^t \langle z_\lambda(s), \mathscr{A} z_\lambda(s)\rangle_{\mathscr{H}}\, ds \\ &+ \int_0^t \operatorname{trace} D^\lambda(z_\lambda(s))(D^\lambda(z_\lambda(s)))^*\, ds \\ &+ 2 \int_0^t \langle z_\lambda(s), D^\lambda(z_\lambda(s))\, d\bar{w}(s)\rangle_{\mathscr{H}}, \end{aligned} \tag{4.12}$$

where

$$\bar{w}(s) = \begin{pmatrix} \beta_1(s) & 0 \\ 0 & \beta_2(s) \end{pmatrix}, \qquad D^\lambda = \lambda R(\lambda, \mathscr{A})\, D,$$

where $D \in \mathscr{L}(\mathscr{H}, \mathscr{L}(K, \mathscr{H}))$ is defined by

$$D(h)\, k = \begin{pmatrix} 0 \\ D_2 h_1 k_{11} + D_3 h_2 k_{22} \end{pmatrix} \tag{4.13}$$

for

$$h = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \in \mathscr{H} \qquad \text{and} \qquad k = \begin{pmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{pmatrix} \in K.$$

Now $D^\lambda \to D$ strongly as $\lambda \to \infty$ and together with (4.11) this implies that the last two terms in (4.12) converge to

$$\int_0^t \operatorname{trace}[D(z(s))(D(z(s)))^*]\, ds + 2\int_0^t \langle z(s), D(z(s))\, d\bar{w}(s)\rangle_{\mathscr{H}} \qquad \text{as} \quad \lambda \to \infty$$

$$= \int_0^t (\|D_2 v\|^2 + \|D_3 v_s\|^2)\, ds + 2\int_0^t \langle v_s, D_2 v\rangle_H\, d\beta_1(s)$$

$$+ 2\int_0^t \langle v_s, D_3 v_s\rangle_H\, d\beta_2(s)$$

from the special form of D and properties of stochastic integrals [4]. Now from the special form of $\mathscr{A}$ from (2.7),

$$2\int_0^t \langle z_\lambda(s), \mathscr{A} z_\lambda(s)\rangle_{\mathscr{H}}\, ds$$

$$= -2\alpha \int_0^t \langle v_s^\lambda, v_s^\lambda\rangle\, ds - 2\int_0^t \langle Fv^\lambda, v_s^\lambda\rangle\, ds, \tag{4.14}$$

where

$$z_\lambda(s) = \begin{pmatrix} v^\lambda \\ v_s^\lambda \end{pmatrix}.$$

Since (4.11) means that $\sup_{0 \leqslant t \leqslant T} E\{\|A^{1/2}(v_\lambda(t) - v(t))\|_H^2\} \to 0$ and $\sup_{0 \leqslant t \leqslant T} E\{\|v_t^\lambda - v_t\|_H^2\} \to 0$ as $\lambda \to \infty$, we can conclude that (4.14) converges to

$$-2\alpha \int_0^t \|v_s\|^2\, ds - 2\int_0^t \langle Fv, v_s\rangle\, ds$$

as $\lambda \to \infty$ in mean square, which together with the remark that $D(\mathscr{A})$ is dense in $\mathscr{H}$ completes the proof.

We may obtain an equation for the second moments by taking expectations of (4.9); thus

$$E\{\|A^{1/2} v\|_H^2 + \|v_t\|_H^2\} - E\{\|A^{1/2} v_0\|^2 + \|v_1\|^2\}$$

$$= -2\alpha \int_0^t E\,\|v_s\|^2 + \int_0^t [E\{\|D_2 v\|^2\}$$

$$+ E\{\|D_3 v_s\|^2\} - 2E\{\langle Fv, v_s\rangle\}]\, ds. \tag{4.15}$$

5. Asymptotic Stability Results

In this section we establish that for our class of second order systems, if we have mean square stability, then the sample paths tend to zero exponentially as $t \to \infty$. Mean square stability for (4.8) means that we have

$$E\{\|z(t)\|_{\mathscr{H}}^2\} \leqslant Ce^{-\gamma t}E\{\|z_0\|_{\mathscr{H}}^2\} \tag{5.1}$$

or equivalently

$$E\{\|A^{1/2}v\|_H^2 + \|v_t\|_H^2\} \leqslant Ce^{-\gamma t}E\{\|A^{1/2}v_1\|_H^2 + \|v_0\|_H^2\} \tag{5.2}$$

for some positive constants $C, \gamma > 0$.

The proof is similar to that in [11] except there it is established for parabolic systems described by the terminology in [16].

LEMMA 5.1. *If the mild solution of* (4.8) *satisfies* (5.1), *then*

$$E\{\sup_{0 \leqslant t < \infty} \|z(t)\|_{\mathscr{H}}^2\} \leqslant C_1 E\{\|z_0\|_{\mathscr{H}}^2\} \tag{5.3}$$

for some constant $C_1 > 0$.

Proof. In the following, $\|F\|$, $\|D\|$ denote norms in $\mathscr{L}(D(A^{1/2}), H)$, $\|D_3\|$ in $\mathscr{L}(H)$, but otherwise $\|\cdot\|$ denotes the norm in H. Now

$$2E\left\{\sup_{0 \leqslant t \leqslant T} \left|\int_0^t \langle v_s, D_2 v\rangle \, d\beta_1(s)\right|\right\}$$

$$\leqslant 6E\sqrt{\int_0^T \langle v_s, D_2 v\rangle^2 \, ds} \qquad \text{by the martingale inequality [8]}$$

$$\leqslant 6E\left\{\sup_{0 \leqslant t \leqslant T} \|v_s\| \sqrt{\int_0^T \|D_2\|^2 \|A^{1/2}v\|^2 \, ds}\right\}$$

$$\leqslant 6\|D_2\| E\left\{\sup_{0 \leqslant t \leqslant T} \|v_s\| \sqrt{\int_0^T \|A^{1/2}v\|^2 \, ds}\right\}$$

$$\leqslant 3\|D_2\| \left[lE\{\sup_{0 \leqslant t \leqslant T} \|v_s\|^2\} + \frac{1}{l} E\left\{\int_0^T \|A^{1/2}v\|^2 \, ds\right\}\right]$$

for arbitrary $l > 0$.

Similarly,

$$2E\left\{\sup_{0\leqslant t\leqslant T}\left|\int_0^t \langle v_s, D_3 v_s\rangle\, d\beta_2(s)\right|\right\}$$
$$\leqslant 3\,\|D_3\|\left[l'E\{\sup_{0\leqslant t\leqslant T}\|v_s\|^2\}+\frac{1}{l'}E\left\{\int_0^T\|v_s\|^2\,ds\right\}\right]$$
$$\text{for arbitrary } l'>0$$

and

$$2E\left\{\sup_{0\leqslant t\leqslant T}\left|\int_0^t \langle Fv, v_s\rangle\, ds\right|\right\}$$
$$\leqslant 2\,\|F\|\,E\left\{\sqrt{\int_0^T\|A^{1/2}v\|^2\,ds}\sqrt{\int_0^T\|v_s\|^2\,ds}\right\}$$
$$\leqslant 2\,\|F\|\left[lE\left\{\int_0^T\|A^{1/2}v\|^2\,ds\right\}+\frac{1}{l}E\left\{\int_0^T\|v_s\|^2\,ds\right\}\right].$$

Thus from (4.9), we deduce the inequality

$$E\{\sup_{0\leqslant t\leqslant T}\|z(s)\|^2_{\mathscr{H}}\}\leqslant E\{\|z_0\|^2_{\mathscr{H}}\}+C_1E\left\{\int_0^T[\|v_s\|^2+\|A^{1/2}v\|^2]\,ds\right\}$$
$$+C_2E\{\sup_{0\leqslant t\leqslant T}\|v_s\|^2\}. \tag{5.4}$$

Now

$$E\left\{\int_0^T(\|v_s\|^2+\|A^{1/2}v\|^2)\,ds\right\}=E\left\{\int_0^T\|z(s)\|^2_{\mathscr{H}}\,ds\right\}$$
$$\leqslant\int_0^T Ce^{-\gamma s}\,ds\,E\{\|z_0\|^2_{\mathscr{H}}\}$$
$$\leqslant\frac{C}{\gamma}E\{\|z_0\|^2_{\mathscr{H}}\}$$

and

$$E\{\sup_{0\leqslant t\leqslant T}\|v_s\|^2\}\leqslant E\{\sup\|z(s)\|^2_{\mathscr{H}}\}$$
$$\leqslant CE\{\|z_0\|^2_{\mathscr{H}}\}$$

and substituting these last inequalities into (5.4) proves the lemma.

LEMMA 5.2. *Under the assumptions of Lemma* 5.1, *there exist positive constants* K *and* b *and a random time* $T(\omega) < \infty$, *such that for* $t > T(\omega)$

$$\|z(t)\|^2_{\mathscr{V}} \leqslant KE\{\|z_0\|^2_{\mathscr{V}}\}\, e^{-bt}. \tag{5.5}$$

Proof. From (4.9), for $t > N$, we have

$$\begin{aligned}
\|z(t)\|^2_{\mathscr{V}} &\leqslant \|z(N)\|^2_{\mathscr{V}} - 2\alpha \int_N^t \|v_s\|^2\, ds + \int_N^t [\|D_2 v\|^2 + \|D_3 v_s\|^2]\, ds \\
&\quad - 2\int_N^t \langle Fv, v_s\rangle\, ds + 2\int_N^t \langle v_s, D_2 v\rangle\, d\beta_1 \\
&\quad + 2\int_N^t \langle v_s, D_3 v_s\rangle\, d\beta_2 \\
&\leqslant \|z(N)\|^2_{\mathscr{V}} + \int_N^t [(2\alpha + \|D_3\|^2)\, \|v_s\|^2 \\
&\quad + \|D_2\|^2\, \|A^{1/2} v\|^2 + 2\, \|F\|\, \|v_s\|\, \|A^{1/2} v\|]\, ds \\
&\quad + 2\left|\int_N^t \langle v_s, D_2 v\rangle\, d\beta_1\right| + 2\left|\int_N^t \langle v_s, D_3 v_s\rangle\, d\beta_2\right| \\
&\leqslant \|z(N)\|^2_{\mathscr{V}} + K_1 \int_N^t \|z(s)\|^2_{\mathscr{V}}\, ds \\
&\quad + 2\left|\int_N^t \langle v_s, D_2 v\rangle\, d\beta_1\right| + 2\left|\int_N^t \langle v_s, D_3 v_s\rangle\, d\beta_2\right|.
\end{aligned}$$

Now

$$\begin{aligned}
&\text{prob}\left\{\sup_{N \leqslant t \leqslant N+1} \left|\int_N^t \langle v_s, D_2 v\rangle\, d\beta_1\right| > \varepsilon\right\} \\
&\quad \leqslant \frac{1}{\varepsilon} E\left\{\sup_{N \leqslant t \leqslant N+1} \left|\int_N^t \langle v_s, D_2 v\rangle\, d\beta_1\right|\right\} \\
&\quad \leqslant \frac{3}{\varepsilon} E \sqrt{\|D_2\|^2 \int_N^{N+1} \|v_s\|^2\, \|A^{1/2} v\|^2\, ds} \\
&\quad \leqslant \frac{3\,\|D_2\|}{\varepsilon} E\left\{\sqrt{\sup_{N \leqslant s \leqslant N+1} \|v_s\|^2}\, \sqrt{\int_N^{N+1} \|A^{1/2} v\|^2\, ds}\right\} \\
&\quad \leqslant \frac{3\,\|D_2\|}{\varepsilon} \left(E\{\sup_{N \leqslant s \leqslant N+1} \|v_s\|^2\}\, E\left\{\int_N^{N+1} \|A^{1/2} v\|^2\, ds\right\}\right)^{1/2} \\
&\quad \leqslant \frac{3\,\|D_2\|}{\varepsilon} (C_1 E\{\|z_0\|^2_{\mathscr{V}}\})^{1/2} \left(\int_N^{N+1} Ce^{-\gamma t} E\, \|z_0\|^2_{\mathscr{V}}\right)^{1/2}
\end{aligned}$$

by (5.1) and Lemma 5.1

$$\leqslant \frac{\text{const}}{\varepsilon} e^{-N\gamma/2} E\{\|z_0\|^2_{\mathscr{V}}\}.$$

Thus

$$
\begin{aligned}
\operatorname{prob}\{ &\sup_{N\leqslant t\leqslant N+1} \|z(t)\|_{\mathscr{H}} \geqslant \delta_N\} \\
&= \operatorname{prob}\{ \sup_{N\leqslant t\leqslant N+1} \|z(t)\|_{\mathscr{H}}^2 \geqslant \delta_N^2\} \\
&\leqslant \operatorname{prob}\left\{ \|z(N)\|_{\mathscr{H}}^2 \geqslant \frac{\delta_N^2}{4} \right\} + \operatorname{prob}\left\{ \int_N^{N+1} \|z(s)\|_{\mathscr{V}}^2\, ds \geqslant \frac{\delta_N^2}{4K_1} \right\} \\
&\quad + \frac{\text{const}}{\delta_N^2}\, e^{-N\gamma/2} E\{\|z_0\|_{\mathscr{H}}^2\} \qquad \text{from (5.6)} \\
&\leqslant \frac{4}{\delta_N^2} E\{\|z(N)\|_{\mathscr{H}}^2\} + \frac{4K_1}{\delta_N^2} \int_N^{N+1} E\{\|z(s)\|_{\mathscr{V}}^2\}\, ds \\
&\quad + \frac{\text{const}}{\delta_N^2}\, e^{-N\gamma/2} E\{\|z_0\|_{\mathscr{H}}^2\} \\
&\leqslant \frac{K_2}{\delta_N^2} E\{\|z_0\|_{\mathscr{H}}^2\}\, e^{-N\gamma} + \frac{K_3}{\delta_N^2}\, e^{-N\gamma/2} E\{\|z_0\|_{\mathscr{H}}^2\} \quad \text{by (5.1)} \\
&\leqslant \frac{K_4}{\delta_N^2}\, e^{-N\gamma/2} E\{\|z_0\|_{\mathscr{H}}^2\}.
\end{aligned}
$$

The Borel Cantelli lemma [8] can now be applied to show that $\exists N(\omega)$ such that for $N > N(\omega)$,

$$\sup_{N\leqslant t\leqslant N+1} \|z(t)\|_{\mathscr{H}}^2 \leqslant K e^{-N\gamma/2} E\{\|z_0\|_{\mathscr{H}}^2\}$$

and (5.5) is established.

Thus in Examples 1 and 2 of Section 3, we deduce that the sample paths tend to zero exponentially fast as $t \to \infty$.

6. Conclusions and Remarks

1. As already remarked by Haussmann in [11], it would be nice to show that the zero solution is pathwise stable w.p.1, but unfortunately this does not seem to be possible. However, using exactly the same proof as that in [11], we can prove in our case, too, that there is asymptotic stability relative to finite dimensional initial conditions, that is, if $z_0 = \binom{v_0}{v_1}$ consists of a finite sum of linear combinations of any orthonormal basis for $\mathscr{H}$.

2. As already mentioned in the Introduction, in [11] a combination of semigroup and variational approaches (from [16]) was used to obtain the

sharpest asymptotic stability results. For the case $F = 0$ in (2.5) or 2.8) the work of Pardoux in [16] yields existence and uniqueness of solutions with continuous sample paths and an energy inequality. These results could probably be extended to the case $F \neq 0$ and replace our Lemmas 2.3 and 4.3. In other words, one could have used a combination of a semigroup approach (to obtain (5.1)) and a variational approach, just as Haussmann did for the parabolic case in [11]. However, one would still need to extend the result of [16] and we have instead chosen a uniform semigroup approach. Essential in the assumptions in [16] is that $-A$ be coercive and it appears that some such restriction on A is necessary to obtain an energy inequality; perhaps the analogue for the semigroup description is that A be dissipative.

3. Although in this paper, we have assumed that the noise occurring via the $z(x, t)$ and $(\partial z/\partial x)(x, t)$ terms is spatially independent, the same approach can be used to prove similar results when this noise is spatially dependent.

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